Numerical Methods

Class- BCA Vth Semester

Topic : Solution of Differential Equations

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Outline

Unit V: Solution of Differential Equations

- Orders of RK method
- Euler's method
- Heun's method
- Runge- Kutta method of 4^{th} order

Introduction

In numerical analysis, the Runge-Kutta method is a family of implicit and explicit iterative methods which includes the Euler's method. RK method refers to a family of one-step method used for numerical solution of IVP. These methods were developed around 1900 by the German mathematician C.Runge and M.W. Kutta

They are all based on the general form of the equation

$$y_{i+1} = y_i + mh$$

Where m is the slope used to construct weight

& h is the interval size.

RK method is known by their orders. For instance, an RK method is called the r-order RK method when slope at r points are used to construct the weighted average slope m.

Orders of RK method

- Euler’s method of order 1
- Heun’s method of order 2
- Bogacki-Shampine method of order 3
- R-K method of order 4
- R-K-Felhberg method of order 4 and 5

Euler’s method

In mathematics and computational science, Euler method (also called forward Euler method) is a first order numerical procedure for solving ordinary differential equation with the given initial value. It is most basic explicit method for numerical integration of ODEs and is simplest Runge-Kutta method. Euler method is named after Leonhard Euler.

$$y_{i+1} = y_i + h(f(x_i, y_i))$$

This formula is known as Euler's Formula and can be used to evaluate $y_1, y_2, \ldots$ of $y(x_1), y(x_2), \ldots$ starting from the initial condition $y_0 = y(x_0)$.

Example: $\frac{dy}{dx} = 3x^2 + 1$ with y(1)=2

Estimate y(2) by Euler's method using h=0.25

Solution: h=0.25

$$y(1) = 2$$

$$y(1.25) = 2 + 0.25[3(1)^2 + 1] = 4.0$$

$$y(1.5) = 3 + 0.25[3(1.25)^2 + 1] = 5.42188$$

$$y(1.75) = 5.42188 + 0.25[3(1.75)^2 + 1] = 7.35938$$

$$y(2.0) = 7.35938 + 0.25[3(1.75)^2 + 1] = 9.90626$$

Advantages/disadvantages of Euler’s methods

Advantages:

- Euler’s method is simple and direct.
- It can be used for nonlinear IVPs.

Disadvantages:

- It is less accurate and numerically unstable.
- Approximation error is proportional to the step size h. Hence good approximation is obtained with a very small h. This requires a large number of time discretization leading to a large computation time.

Heun's method

In mathematics and computational science , Heun's method may refer to improved or modified Euler's method or a similar two-stage Runge-Kutta method. The Heun's method is :

$$y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, (y_{i+1})^e)$$

Where $(y_{i+1})^e = y_i + h(f(x_i, y_i))$

Example:$\frac{dy}{dx} = 3x^2 + 1$ **with y(1)=2**

Solution: iteration 1:$m_1 = \frac{2\times2}{1} = 4.0$

$$y_e(1.25) = 2 + 0.25(4.0) = 3.0$$

$m_2 = \frac{2\times3.0}{1.25} = 4.8$

$$y(1.25) = 2 + \frac{0.25}{2}(4.0 + 4.8) = 3.1$$

Iteration2: $m_1 = \frac{2\times 3.1}{1.25} = 4.96$

$$y_e(1.5) = 3.1 + 0.25(4.96) = 4.34$$

$m_2 = \frac{2\times 4.34}{1.5} = \mathbf{4.8}$

$$y(1.5) = 3.1 + \frac{0.25}{2}(4.96 + 5.79) = 4.44$$

Iteration3: $m_1 = \frac{2\times 4.44}{1.5} = 5.92$

$$y_e(1.75) = 4.44 + 0.25(5.92) = 5.92$$

$m_2 = \frac{2\times 5.92}{1.75} = \mathbf{6.77}$

$$y(1.75) = 4.44 + \frac{0.25}{2}(5.92 + 6.77) = 6.03$$

Iteration4: $m_1 = \frac{2\times 6.03}{1.75} = 6.89$

$$y_e(2.0) = 6.03 + 0.25(6.89) = 7.75$$

$m_2 = \frac{2\times 7.75}{2} = \mathbf{7.75}$

$$y(2.0) = 6.03 + \frac{0.25}{2}(6.89 + 7.75) = 7.86$$

Advantages/ Disadvantages of Heun's method

Advatages:

- It is accurate for numerical problems.
- It requires less error then Euler's method . Hence good approximation is obtained with a very small h.

Disadvantages:

- It is not used for more than third order.

Bogacki-Shampine method

The Bogacki-Shampine method is a method for the numerical solution of ordinary differential equations, that was proposed by Przemyslaw Bogacki and Lawrence F. Shampine in 1989. This method is the Runge-Kutta of order three.

The standard notations, the differential equation to be solved is $y' = f(t, y)$. where y_n denotes the numerical solution at timet_n and h_n is the step size, defined by $h_n = t_{n-1} - t_n$.

Then one step Bogacki-Shampine method is given by

$$z_{n+1} = y_n + \frac{7}{24}hk_1 + \frac{1}{4}hk_2 + \frac{1}{3}hk_3 + \frac{1}{8}hk_4$$

Where $k_1 = f(x_n, y_n)$

$$k_2 = f\left(x_n + \frac{h}{2}, y_n + k_1\frac{h}{2}\right)$$

$$k_3 = f\left(x_n + \frac{3h}{4}, y_n + \frac{3k_2h}{2}\right)$$

$$k_4 = f(x_n + h, y_{n+1})$$

y_{n+1}=$y_n + \frac{2}{9}hk_1 + \frac{1}{3}hk_2 + \frac{4}{9}hk_3$

The difference between y_{n+1} and z_{n+1} can be used to adapt the step size.

Examples: $\frac{dy}{dx} = 3x^2 + 1$ with $y(1) = 2$

Solution: here $h = 0.5$

Iteration 1: $k_1 = f(x_0, y_0) = f(1,2)$

$$k_1 = 3(1)^2 + 1 = 4$$

$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + k_1\frac{h}{2}\right) = f\left(1 + \frac{0.5}{2}, 2 + (4)\frac{0.5}{2}\right)$$

$$= f(1.25,3) = 5.6875$$

$$k_3 = f\left(x_0 + \frac{3h}{4}, y_0 + \frac{3k_2h}{2}\right) = f(1.375,6.2656) = 6.6718$$

$$y_1 = y_0 + \frac{2}{9}hk_1 + \frac{1}{3}hk_2 + \frac{4}{9}hk_3 = 4.8749$$

$$k_4 = f(x_0 + h, y_1) = f(1.5,2) = 7.75$$

$$z_1 = y_0 + \frac{7}{24}hk_1 + \frac{1}{4}hk_2 + \frac{1}{3}hk_3 + \frac{1}{8}hk_4 = 4.8906$$

Iteration 2: $k_1 = f(x_1, y_1) = f(1.5, 4.8749)$

$$k_1 = 3(1.5)^2 + 1 = 7.75$$

$$k_2 = f(1.75, 5.2499) = 10.187$$

$$k_3 = f(1.875, 12.515) = 11.5468$$

$$y_2 = y_1 + \frac{2}{9}hk_1 + \frac{1}{3}hk_2 + \frac{4}{9}hk_3 = 9.9997$$

$$k_4 = f(x_1 + h, y_2) = f(2, 9.9997) = 13$$

$$z_2 = y_1 + \frac{7}{24}hk_1 + \frac{1}{4}hk_2 + \frac{1}{3}hk_3 + \frac{1}{8}hk_4$$

$$= 10.0154$$

Iteration 3: $k_1 = f(x_2, y_2) = f(2{,}9.9997)$

$$k_1 = 3(2)^2 + 1 = 13$$

$$k_2 = f(2.25{,}13.2497) = 16.1875$$

$$k_3 = f(2.375{,}22.1403) = 17.9218$$

$$y_3 = y_2 + \tfrac{2}{9}hk_1 + \tfrac{1}{3}hk_2 + \tfrac{4}{9}hk_3 = 18.12464$$

$$k_4 = f(x_1 + h, y_3) = f(3{,}18.12464) = 28$$

$$z_3 = y_2 + \frac{7}{24}hk_1 + \frac{1}{4}hk_2 + \frac{1}{3}hk_3 + \frac{1}{8}hk_4$$

$$= 17.03087$$

Advantages/Disadvantages of Bogacki-Shampine method

Advantages:

- Used for approximately three functions evaluations per steps.
- Low order method is more suitable.

Disadvantages:

- Large order method is not suitable for ODEs.
- This requires a large number of time discretization leading to a large computation time.

Runge- Kutta method of 4th order

In numerical analysis, the **Runge–Kutta methods** are a family of implicit and explicit iterative methods, which include the well-known routine called the Euler Method, used in temporal discretization for the approximate solutions of ordinary differential equations.[1] These methods were developed around 1900 by the German mathematicians C. Runge and M. W. Kutta.

Runge-Kutta of 4th order is given by

$$y_{i+1} = y_i + \left(\frac{m_1+2m_2+2m_3+m_4}{6}\right)h$$

Where $m_1 = f(x_i, y_i)$

$$m_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{m_1 h}{2}\right)$$

$$m_3 = f\left(x_i + \frac{h}{2}, y_i + \frac{m_2 h}{2}\right)$$

$$m_4 = f(x_i + h, y_i + m_3 h)$$

Example: use the RK method to estimate $y(0.4)$ when $y'(x) = x^2 + y^2$ with $y(0) = 0$.

Assume $h = 0.2$

$$f(x+y) = x^2 + y^2$$

Solution: $m_1 = f(x_0, y_0) = {x_0}^2 + {y_0}^2 = 0$

$$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2}\right) = f(0.1,0) = (0.1^2 + 0^2)$$

$$= 0.01$$

$$m_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_2 h}{2}\right) = \left(\frac{0.2}{2}, \frac{(0.01)(0.2)}{2}\right)$$

$$= 0.01$$

$$m_4 = f(x_0 + h, y_0 + m_3 h) = f(0.2, (0.01)(0.2) = 0.04$$

$$y(0.2) = 0 + \left(\frac{0 + 2(0.01) + 2(0.01) + 0.04}{6}\right)(0.2)$$

$$= 0.002667$$

Iteration2: $x_1 = 0.2 \; and \; y_1 = 0.002667$

$$m_1 = f(x_1, y_1) = {x_1}^2 + {y_1}^2 = 0.04$$

$$m_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{m_1 h}{2}\right) = 0.090044$$

$$m_3 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{m_2 h}{2}\right) = 0.090136$$

$$m_4 = f(x_1 + h, y_1 + m_3 h) = 0.160428$$

$$y(0.4) = 0.002667 + \left(\frac{0.04 + 2(0.090044) + 2(0.090136) + 0.16028}{6}\right)(0.2) = 0.021360$$

Advantages/Disadvantages of Runge-Kutta 4th order

Advantages:

It is used for temporal discretization of ordinary differential equation.

Disadvantages:

Runge-Kutta method are generally unsuitable for the solution of stiff equations.

It increase the number of iteration steps.

Runge-Kutta-Fehlberg(4,5) method

- In mathemtics, the **Runge–Kutta–Fehlberg method** (or **Fehlberg method**) is an algorithm in numerical analysis for the numerical solution of ordinary differential equations. It was developed by the German mathematician Erwin Fehlberg and is based on the large class of Runge–Kutta methods.
- The novelty of Fehlberg's method is that it is an embedded method from the Runge-Kutta family, meaning that identical function evaluations are used in conjunction with each other to create methods of varying order and similar error constants. The method presented in Fehlberg's 1969 paper has been dubbed the **RKF45** method, and is a method of order 4 with an error estimator of order 5.

Each step requires the use of following six values:

$$k_1 = hf(x_i, y_i)$$

$$k_2 = hf\left(x_i + \frac{h}{4}, y_i + \frac{k_1}{4}\right)$$

$$k_3 = hf\left(x_i + \frac{3h}{8}, y_i + \frac{3k_1}{32} + \frac{9k_2}{32}\right)$$

$$k_4 = h\,f\left(x_i + \frac{12h}{32}, y_i + \frac{1932k_1}{2197} - \frac{7200k_2}{2197} + \frac{7296k_3}{2197}\right)$$

$$k_5 = h\,f\left(x_i + h, y_i + \frac{439k_1}{216} - 8k_2 + \frac{3680k_3}{513} - \frac{845k_4}{4104}\right)$$

$$k_6 = h\,f\left(x_i + \frac{1}{2}h, y_i - \frac{8k_1}{27} + 2k_2 + \frac{3544k_3}{2565} + \frac{1859k_4}{4104} - \frac{11k_5}{40}\right)$$

then the approximation to the solution of the I.V.P. is made using a Runge-Kutta method of order 4.

$$y_{i+1} = y_i + \frac{25k_1}{216} + \frac{4108k_3}{2565} + \frac{2197k_4}{4104} - \frac{1k_5}{5}$$

A better value for the solution is determined using a Runge-Kutta method of order 5:

$$z_{i+1} = y_i + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6$$

Exercise

1. Apply Runge-Kutta method find the solution of the differential equation $\frac{dy}{dx} = 3x + \frac{1}{2}y$ with $y_0 = 1$ at $x = 0.1$ **[Ans.** 1.066652421875]

2. Given $\frac{dy}{dx} = 1 + y^2$ where $y = 0$ when $x = 0$, find $y_{(0.2)}$, $y_{(0.4)}$ and $y_{(0.6)}$, using Runge-Kutta formula of order four. **[Ans.** $y_{(0.2)} = 0.2027$, $y_{(0.4)} = 0.4228$, $y_{(0.6)} = 0.6841$]

3. Use classical Runge-Kutta method of fourth order to find the numerical solution at $x = 1.4$ for $\frac{dy}{dx} = y^2 + x^2, y(1) = 0$. Assume step size $h = 0.2$. **[Ans.** $y(1.2)= 0.246326$, $y(1.4)=0.622751489$]

4. Using Runge-Kutta method to solve $10\frac{dy}{dx} = x^2 + y^2$ $y_{(0)} =$ for the interval $0 < x \leq 0.4$ with $h = 0.1$. **[Ans.** 1.0101, 1.0207, 1.0318, 1.0438]

5. Solve the differential equation $\frac{dy}{dx} = \frac{2x-1}{x^2}y + 1$ where $x_0 = 1$, $y_0 = 2$, $h = 0.2$. Obtain $y_{(1.2)}$ and $y_{(1.4)}$ using Runge-Kutta method. **[Ans.** 2.658913 and 3.432851]

6. Using Runge-Kutta method solve simultaneous differential equation $\frac{dy}{dx} = f(x, y, t) = xy + t$ and $\frac{dy}{dx} = ty + x = g(x, y, t)$ where $t_0 = 0$, $x_0 = 1$, $y_0 = -1$, $h = 0.2$. **[Ans.** $y_{(0.2)} = -0.8341$]

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Thanks