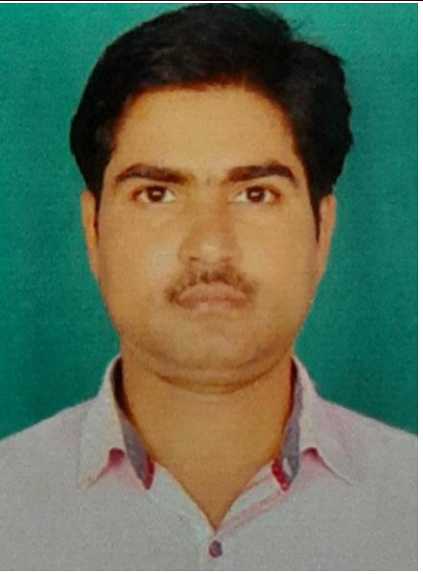
Mathematics III
Class-BCA IV Semester



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OUTLINE-

UNIT :- IV

FOURIER SERIES

Equation, Condition, value of variable

Examples

As we know that TAYLOR SERIES representation of functions are valid only for those functions which are continuous and differentiable. But there are many discontinuous periodic function which requires to express in terms of an infinite series containing 'sine' and 'cosine' terms.

FOURIER SERIES, which is an infinite series representation of such functions in terms of 'sine' and 'cosine' terms, is useful here. Thus, FOURIER SERIES, are in certain sense, more UNIVERSAL than TAYLOR's SERIES as it applies to all continuous, periodic functions and also to the functions which are discontinuous in their values and derivatives. FOURIER SERIES a very powerful method to solve ordinary and partial differential equation, particularly with periodic functions appearing as non-homogenous terms.

FOURIER SERIES can be generally written as,

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx),$$

Where,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad \text{......... (1.1)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \quad \text{......... (1.2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \quad \text{......... (1.3)}$$

Fourier series make use of the orthogonality relationships of the sine and cosine functions.

BASIS FORMULAE OF FOURIER SERIES

The Fourier series of a periodic function $f(x)$ with period 2π is defined as the trigonometric series with the coefficient a0, an and bn, known as *FOURIER COEFFICIENTS*, **determined by formulae (1.1), (1.2) and (1.3).**

The individual terms in Fourier Series are known as *HARMONICS.*

Every function $f(x)$ of period 2π satisfying following conditions known as *DIRICHLET'S CONDITIONS*, **can be expressed in the form of Fourier series.**

CONDITIONS :-

1. f(x) is bounded and single value.

(A function f(x) is called single valued if each point in the domain, it has unique value in the range.)

2. f(x) has at most, a finite no. of maxima and minima in the interval.
3. f(x) has at most, a finite no. of discontinuities in the interval.

EXAMPLE:

$\sin^{-1}x$, we can say that the function $\sin^{-1}x$ cant be expressed as Fourier series as it is not a single valued function.

tanx, also in the interval (0,2π) cannot be expressed as a Fourier Series because it is infinite at x= π/2.

FOURIER SERIES FOR EVEN AND ODD FUNCTIONS

EVEN FUNCTIONS

If function f(x) is an even periodic function with the period 2L (–L ≤ x ≤ L), then f(x)cos(nπx/L) is even while f(x)sin(nπx/L) is odd.

Thus the Fourier series expansion of an even periodic function f(x) with period 2L (–L ≤ x ≤ L) is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{\pi n x}{L}$$

Where, $$a_0 = \frac{2}{L}\int_0^L f(x)dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \qquad n = 1,2,\cdots$$

$$b_n = 0$$

ODD FUNCTIONS

If function f(x) is an even periodic function with the period 2L (–L ≤ x ≤ L), then f(x)cos(n□x/L) is even while f(x)sin(n□x/L) is odd.

Thus the Fourier series expansion of an odd periodic function f(x) with period 2L (–L ≤ x ≤ L) is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L})$$

Where, $$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \qquad n = 1, 2, \cdots$$

EXAMPLES..

Question.: Find the fourier series of $f(x) = x^2+x$, $- \leq x \leq$.

Solution.: The fourier series of $f(x)$ is given by,

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx),$$

Using above,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x^2 + x)dx$$

$$= \frac{1}{\pi}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^3}{3} + \frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2}\right) = \frac{2\pi^3}{3} = a_0$$

Now,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x^2 + x)\cos nx dx$$

$$= \frac{1}{\pi}\left[(x^2 + x)\left(\frac{\sin nx}{n}\right) - (2x+1)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(2\pi+1)\frac{\cos n\pi}{n^2} - (-2\pi+1)\frac{\cos n\pi}{n^2}\right]$$

$$= \frac{1}{\pi}\left[(2\pi+1)\frac{(-1)^n}{n^2} - (-2\pi+1)\frac{(-1)^n}{n^2}\right]$$

$$= \frac{4(-1)^n}{n^2}$$

Now,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x^2 + x)\sin nx dx$$

$$= \frac{1}{\pi}\left[(x^2 + x)\left(-\frac{\cos nx}{n}\right) - (2x+1)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{(\pi^2 + \pi)}{n}(-1)^n + \frac{(\pi^2 + \pi)}{n}(-1)^n \right]$$

$$= \frac{(-1)^n}{\pi n}\left[-\pi^2 - \pi + \pi^2 - \pi\right]$$

$$= -\frac{2(-1)^n}{n}$$

Hence fourier series of, f(x) = x^2+x,

$$x^2 + x = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4(-1)^n}{n^2}\cos nx - \frac{2(-1)^n}{n}\sin nx \right]$$

QUESTIONS:-

Find the Fourier series for the square 2π-periodic wave defined on the interval $[-\pi, \pi]$:

$$f(x) = \begin{cases} 0, & \text{if} \quad -\pi \leq x \leq 0 \\ 1, & \text{if} \quad 0 < x \leq \pi \end{cases} .$$

Find the Fourier series for the triangle wave

$$f(x) = \begin{cases} \frac{\pi}{2} + x, & \text{if} \quad -\pi \leq x \leq 0 \\ \frac{\pi}{2} - x, & \text{if} \quad 0 < x \leq \pi \end{cases} ,$$

defined on the interval $[-\pi, \pi]$.

Reference Books:

1. A.B. Mathur and V.P. Jaggi, "Advanced Engineering Mathematics", Khanna Publishers, 1999.

2. 2. H.K. Dass, "Advanced Engineering Mathematics", S. Chand & Co., 9th Revised Ed.

Declaration

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Thanks